

A cost operator approach to multistage location-allocation

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The multistage factory-warehouse location-allocation problem is to decide on locations of warehouses and shipping amounts from the factories through the warehouses to meet customer demands in such a way that the total fixed plus variable costs are minimized. Capacity constraints at factories and warehouses are also imposed.

We present a cost operator algorithm for solving this problem. The algorithm takes into account the network structure and the submodularity of the objective function. Computational results with problems taken from the literature as well as new problems are provided.

Keywords: Location, location-allocation, mixed integer programming, submodularity, cost operator, network location

1. Introduction

In this paper we present a cost operator algorithm for solving multistage factory warehouse location-allocation problems. The decision variables correspond to the warehouse locations and the shipping amounts from the factories through the warehouses and into the demand centers. The problem is to minimize the total fixed costs of locating warehouses plus the total variable shipping cost, subject to possible capacity

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restrictions at the factories and the warehouses, plus the demand requirements at the demand centers.

The algorithm takes advantage of the network structure of the supply and demand constraints and the submodularity of the objective function. We use cost operators [23], to facilitate the movement up and down the search tree. This greatly reduces the amount of time spent solving transportation subproblems, which frequently comprises up to 90% of the computational burden.

In Section 2 we give a problem formulation and we discuss some of the more recent research related to the multistage location problem. In Section 3 we point out lower bounds and fathoming rules obtainable from submodular set functions. In Section 4 we transform the multistage location problem into a transportation problem format, and we show how cost operators can be used to generate feasible solutions. Section 5 contains a description of the cost operator branch and bound algorithm. In Section 6 we give an example, and in Section 7 we provide extensive computational experience on problems from the literature plus some multistage problems of our own.

2. Problem formulation

We describe the location problem to be studied in this paper as well as similar models which have been presented in the literature. We use the following notation:

q = number of factories; $I' = \{1, \dots, q\}$
 l = number of warehouses; $I = \{q + 1, \dots, m\}$;
 r = number of demand centers;
 $m = q + l + 1$; $n = l + r + 1$; $J' = \{1, \dots, l\}$,
 $J = \{l + 1, \dots, n\}$. Note that m and n have each been increased by 1 so that a dummy factory and a slack demand center can be introduced; see Section 4.

f_i = fixed cost of opening warehouse i ;
 c_{ij} = $\begin{cases} \text{cost per unit} & \begin{cases} \text{factory } i \text{ to} & i \in I' \\ \text{warehouse } j & j \in J' \end{cases} \\ \text{of distance} & \begin{cases} \text{factory } i \text{ to} & i \in I' \\ \text{demand center } j & j \in J \end{cases} \\ \text{shipping from} & \begin{cases} \text{warehouse } i \text{ to} & i \in I \\ \text{demand center } j & j \in J \end{cases} \end{cases}$

- A_i = capacity of factory i ;
 S_i = capacity of warehouse i ;
 d_j = demand at location j ;
 x_{ij} = amount shipped from location i to location j ;
 $y_i = \begin{cases} 1 & \text{if warehouse } i \text{ is open,} \\ 0 & \text{otherwise.} \end{cases}$

The multistage or intermediate location problem, which we will call Problem P, can be formulated as:

$$\text{Minimize } Z(T) = \sum_{T \subseteq I} \sum_{i \in I' \cup T} \sum_{j \in J' \cup J} c_{ij} x_{ij} + \sum_{i \in T} f_i y_i, \quad (1)$$

subject to

$$\sum_{j \in J' \cup J} x_{ij} \leq A_i, \quad i \in I'; \quad (2)$$

$$\sum_{j \in J} x_{ij} - S_i y_i \leq 0, \quad i \in T; \quad (3)$$

$$\sum_{i \in I' \cup T} x_{ij} \geq d_j, \quad j \in J; \quad (4)$$

$$\sum_{h \in I'} x_{hi} - \sum_{j \in J} x_{ij} = 0, \quad i \in T; \quad (5)$$

$$x_{ij} \geq 0, \quad i \in I' \cup I, \quad j \in J' \cup J; \quad (6)$$

$$y_i = \begin{cases} 1, & i \in T, \\ 0, & \text{otherwise.} \end{cases} \quad (7)$$

Constraints (2) and (3) ensure that the amount shipped out of a factory or a warehouse should not exceed its capacity. Constraints (4) require that the demand at each demand center be satisfied. Constraints (5) are the standard 'conservation of flow' constraints which require that the amount shipped into each warehouse equals the amount shipped out. Constraints (6) and (7) are the nonnegativity and integrality constraints respectively. The objective in problem P is to minimize the total transportation costs from shipping plus the total fixed costs of opening warehouses, while satisfying the customer demand. When

$$S_i < \sum_{j \in J} d_j \quad (8)$$

we say that warehouse i is *capacitated* since it cannot satisfy all of the demand by itself. The factories can also be either capacitated or uncapacitated depending upon the size of A_i . Note that in the multistage formulation when $I' \neq \emptyset$, the total warehouse capacity need not exceed the total demand since units can also be shipped directly from the factories to the demand centers.

The literature on location problems is enormous.

In this section we discuss recent literature only for those problems which are closely related to P. Research on multistage location problems has been limited. Geofrion and Graves [12] described and tested an algorithm for solving multistage multicommodity distribution systems using Benders' decomposition. Ellwein and Gray [8] described and tested an algorithm for solving single stage ($I' = \emptyset$) location problems and proposed, but did not test, an algorithm for solving multistage problems. Most of the papers in the literature have concentrated on problems which are closely related to the single stage version of P, which we denote by P1. Early methods attempt to solve P1 by imbedding a linear programming relaxation of P1 into an implicit enumeration scheme, see [1,6,7,19,20,21]. Lagrangean relaxations have also been used to solve P1, see [4,5,11,16] with some excellent computational results being reported by Nauss [17].

Recently 'dual ascent' methods which utilize properties of the dual to various linear programming relaxations of P1 have been used successfully to solve uncapacitated location problems [3,9]. Also Guignard and Spielberg have generalized the dual ascent methods in [3,9] to handle capacitated location problems.

In the following sections we will describe an algorithm for solving P which does not solve a relaxation of P. Instead we make use of some lower bounds obtainable from the submodular property of Z , together with some fathoming rules, to enumerate implicitly the solutions to P. The movement in the search tree from one feasible solution to another is facilitated by applying cost operators [22] to P, which significantly reduces the amount of time spent in solving transportation sub-problems. In the next section we discuss lower bounds on the value of Z which are used in the algorithm in Section 5.

3. Objective function lower bounds

Many of the lower bound properties which we present here have previously been discussed in [2,10,22], in the context of the simple uncapacitated location problem. These lower bounds and their properties are also useful in solving other kinds of mathematical programming problems.

There are many properties which can be used to define submodular set functions. For a discussion of them see [10] or [18]. To define one such function let Z be a real valued function defined on the finite set of subsets of I .

For notational convenience we define:

$$\Delta Z_B(A) = Z(A \cup B) - Z(A), \quad A \subseteq I, \quad B \subseteq I. \quad (9)$$

Then any of the following properties can be used to define a submodular function:

(i) $Z(A \cup B) + Z(A \cap B) \geq Z(A) + Z(B)$ for all subsets $A, B \subseteq I$.

(ii) $\Delta Z_i(A) \leq \Delta Z_i(B) \quad \forall A \subseteq B \subseteq I, \quad i \in I - B$. (10)

(iii) Let $\{B_1, \dots, B_r\}$ be a partition of $B - A$; then

$$Z(A) \geq Z(B) - \sum_{k=1}^r \Delta Z_{B_k}(B - B_k), \quad A \subseteq B \subseteq I. \quad (11)$$

In most expositions property (i) is taken to be the definition of submodularity, and properties (ii) and (iii) are shown to be equivalent. Details are omitted.

In the context of this paper, $Z(A)$ represents (1). The fact that $Z(A)$ is submodular for assignment problems was proved by Shapley [21]. (Shapley remarks that the extension to transportation problems is straightforward.) Note that in P, $Z(\emptyset)$ represents the value of the solution where all of the shipments originate from the factories. When $I' = \emptyset$, as in P1, one must be careful in defining $Z(A)$ if A happens to be an infeasible set of warehouses, i.e., when $\sum_{i \in A} S_i < \sum_{j \in J} d_j$. In this case we define a 'dummy factory' which is always available to service the demand centers at a high shipping cost. This makes $Z(A)$ large enough so that the value of $Z(A)$ for any infeasible A is at least as large as the value of any feasible solution. The dummy factory approach preserves the submodularity of Z and also yields a different value of Z for solutions having different degrees of infeasibility. As we will see later, this is helpful in deciding which warehouses to open when we are working with infeasible sets of warehouses.

Properties (10) and (11) can be interpreted as adding or subtracting warehouse from a given set A of open warehouses. For example property (10) says that the addition of warehouse i to the set A decreases the total cost by at least as much as the addition of warehouse i to the set B when $A \subseteq B$. Thus it is similar to the 'decreasing returns to scale' condition in economics.

Properties (10) and (11) can be used to characterize solutions to P and to derive lower bounds on the value of any solution to P. The algorithm to be described in Section 5 searches for sets $T \subseteq I$ which have the following two properties:

(i) $\Delta Z_i(T) \geq 0, \quad \forall i \in I - T,$

(ii) $\Delta Z_i(T - \{i\}) \leq 0, \quad \forall i \in T.$

Sets $T \subseteq I$ which satisfy (i) and (ii) are such that the addition to T or the deletion from T of a single facility does not cause Z to decrease. Clearly any optimal solution to P satisfies (i) and (ii). In a sense properties (i) and (ii) characterize the set of all 'locally optimal' solutions to P. The globally optimal solution is the best local solution, which must be found by a search process. One of the factors which make it hard to find the globally optimal solution is that there are many locally optimal solutions which have nearly optimal objective function values. Thus in any enumeration procedure, a considerable amount of effort is normally required to eliminate these nearly optimal solutions from consideration. In a practical sense however, it may be true that the nearly optimal solutions, say those within 1% of optimality, may indeed be as valuable or 'as optimal' to a decision maker as a globally optimal solution. Given the inaccuracies in the cost data and other environmental and political factors which must be taken into consideration, it would be desirable to have not only an optimal solution to P, but in addition a list of solutions which are nearly optimal. We will point out how this may be accomplished using the algorithm of Section 5.

In the initialization phase of any location algorithm, two rules can be applied in order to permanently open or close warehouses.

Rule 1. If $\Delta Z_i(I - \{i\}) \leq 0$ for all $i \in I^*$ for some subset $I^* \subseteq I$, then there exists an optimal solution with all warehouses $i \in I^*$ open.

Proof. Let T be an optimal set of warehouses and assume that $T - I^* = \{i_1, \dots, i_q\} \neq \emptyset$. Define $T_0 = T$ and $T_k = T_{k-1} \cup \{i_k\}$, $k = 1, \dots, q$. Then by property (10) we have

$$\Delta Z_{i_k}(T_{k-1}) = Z(T_k) - Z(T_{k-1}) \leq \Delta Z_{i_k}(I - \{i_k\}) \leq 0, \quad k = 1, \dots, q$$

and thus $T \cup I^*$ is also an optimal set of warehouses.

Rule 2. If $\Delta Z_i(I^*) \geq 0$ for all $i \in C^*$ for some $C^* \subseteq I - I^*$, there exists an optimal solution with all warehouses $i \in C^*$ closed.

The justification for Rule 2 is similar to that for Rule 1. We will show, in the next section how the testing of Rule 1 and Rule 2 requires only the applica-

tion of two cost operators for each warehouse. In many cases, as will be seen in Section 7, Rules 1 and 2 can be used to fix open and closed a large portion of the warehouses in an optimal solution.

Another property of submodular set functions which can be derived from (10) and (11) is the following: Let $\{A_1, \dots, A_r\}$ and $\{Q_1, \dots, Q_t\}$ be partitions of $A - T$ such that $Q_i \subseteq A_j$ for each $i = 1, \dots, t$ and some $j = 1, \dots, r$. Then

$$\begin{aligned} Z(T) &\geq Z(A) - \sum_{k=1}^r \Delta Z_{A_k}(A - A_k) \\ &\geq Z(A) - \sum_{k=1}^t \Delta Z_{Q_k}(A - Q_k) \end{aligned} \quad (12)$$

for all subsets T satisfying $T \subseteq A \subseteq I$. The quantity on the right-hand side of (11) provides a value of $Z(T)$, for any $T \subseteq I$. Property (12) says when we use a more refined partition of $A - T$, we get a weaker lower bound. Suppose we let $A = I$. Then the most refined partition of $I - T$ is clearly $\{\{i_1\}, \dots, \{i_t\}\}$, where $i_k \in I - T$ and $t = |I - T|$. Then for this partition (12) yields:

$$Z(T) \geq Z(I) - \sum_{k=1}^t \Delta Z_{i_k}(I - \{i_k\}). \quad (13)$$

Among the class of lower bounds (11), (13) is the weakest. Notice that once Rule 1 has been applied all of the terms on the right-hand side of (13) have been calculated. Thus (13) can be applied at any time after Rule 1 has been tested without any extra computational effort. A stronger lower bound than (13) could be obtained with some added computational effort by partitioning I into sets, A_k , of size two and calculating $\Delta Z_{A_k}(I - A_k)$. We have not yet tested this idea.

Another fathoming device which provides an upper bound on the maximum number of warehouses in an optimal solution to P can be obtained as follows: Let $t = |I - I^*|$. Let $f_{k_1}, \dots, f_{k_t}$, $k_i \in I - I^*$, be a nondecreasing ordering of the fixed charges not in I^* . Define

$$DV_{j_i} = \Delta Z_{j_i}(I - \{j_i\}) - f_{j_i}, \quad j_i \in I - I^* \quad (14)$$

and let $DV_{j_1}, \dots, DV_{j_t}$ be nonincreasing ordering of the DV_{j_i} . (Note that $DV_{j_i} \leq 0$.) The quantity $-DV_{j_i}$ represents the smallest possible incremental savings on the total shipping cost when warehouse j_i is opened. Let Z^u be any upper bound on the optimal objective func-

tion value for P. Then an upper bound on the number of warehouses in an optimal solution to P (containing I^*) is $|I^*| + k^*$, where

$$\begin{aligned} Z(I) - \sum_{l \in I} f_l + \sum_{l=1}^{k^*} f_{k_l} + \sum_{s=1}^{t-k^*} |DV_{j_s}| &\leq \\ &\leq Z^u \leq Z(I) - \sum_{l \in I} f_l + \sum_{l=1}^{k^*+1} f_{k_l} + \sum_{s=1}^{t-k^*-1} |DV_{j_s}|. \end{aligned} \quad (15)$$

Note that the first two terms on the right-hand side of (15) give the smallest possible shipping cost, the next term is the smallest sum of $k^* + 1$ fixed charges, and the last term is the smallest possible cost of closing down $t - (k^* + 1)$ warehouses. Therefore the right-hand side of (15) is a lower bound on the value of an optimal solution to P containing $|I^*| + k^* + 1$ warehouses. If this lower bound exceeds Z^u , then any optimal solution will contain at most $|I^*| + k^*$ open warehouses.

In the algorithm of Section 5 we use as fathoming devices bounds obtained from (13) on the value of an optimal solution, and bounds obtained from (15) on the number of warehouses in an optimal solution.

4. Use of cost operators to solve problem P

Given the choice of a subset $T \subseteq I$, problem P becomes an ordinary transshipment problem. Rather than resolving this problem each time T changes, we use the operator theory of parametric programming [23] to change the problem and derive the new optimal solution simultaneously.

We shall say that cell (i, j) has been *fixed out* of the basis when a cell cost operator has been applied to the problem and its solution so that the cost c_{ij} has been driven to $+M$, where M is so large that $x_{ij} = 0$ in any optimal solution to the new problem.

We also say that cell (i, j) has been *fixed in* the basis when a cell cost operator has been applied to the problem and its solution so that the cost c_{ij} has been driven to $-M$, where M is so large that $x_{ij} = \min(S_i, d_j)$ in any optimal solution to the new problem.

Fig. 1 shows an example of a $2 \times 3 \times 4$ multistage location problem. Rows 1 and 2 correspond to the factories, rows 3, 4, and 5 to the warehouses, and row 6 represents the dummy factory. Columns 1, 2 and 3 correspond to the warehouses, columns 4 through 7 to the demand centers, and column 8 is a slack

		Warehouses			Demand Centers				
Factories	{	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	c_{16}	c_{17}	A_1
		c_{21}	c_{22}	c_{23}	c_{24}	c_{25}	c_{26}	c_{27}	A_2
Warehouses	{	$\frac{f_3}{S_3}$	M	M	$c_{34} + \frac{f_3}{S_3}$	$c_{35} + \frac{f_3}{S_3}$	$c_{36} + \frac{f_3}{S_3}$	$c_{37} + \frac{f_3}{S_3}$	S_3
		M	$\frac{f_4}{S_4}$	M	$c_{44} + \frac{f_4}{S_4}$	$c_{45} + \frac{f_4}{S_4}$	$c_{46} + \frac{f_4}{S_4}$	$c_{47} + \frac{f_4}{S_4}$	S_4
		M	M	$\frac{f_5}{S_5}$	$c_{54} + \frac{f_5}{S_5}$	$c_{55} + \frac{f_5}{S_5}$	$c_{56} + \frac{f_5}{S_5}$	$c_{57} + \frac{f_5}{S_5}$	S_5
Dummy Factory		$-M$	M	M	M	M	M	M	$\sum_{i \in I} S_i$
		S_3	S_4	S_5	d_4	d_5	d_6	d_7	d_8

Fig. 1.

column. Rows 1 and 2 contain the costs of shipping from the factories to the warehouses and from the factories directly to the demand centers. Cells (3, 1), (4, 2), and (5, 3) contain the fixed charge of the corresponding warehouse, divided by its capacity. In any solution to P, these cells will contain the unused warehouse capacity. Cells (3, 2), (3, 3), (4, 1), (4, 3), (5, 1) and (5, 2) cannot be used because of their large costs; in effect these arcs have been removed from the problem. The cells in rows 3 through 5, and columns 4 through 7, contain the costs of shipping from each of the warehouses to each of the demand centers plus the proportional fixed charges. Notice that the cells in dummy factory row 6 and slack column 8 contain costs equal to M or $-M$. This can be explained in the following manner. To obtain the solution, where:

$$y_i = 0 \quad \text{set } c_{in} = -M \text{ and } c_{m,i-q} = -M;$$

$$y_i = 1 \quad \text{set } c_{in} = M \text{ and } c_{m,i-q} = M$$

(in Fig. 1, $m = 6$, $n = 8$, $q = 2$, and $i \in \{3, 4, 5\}$).

By solving the transportation problem in Fig. 1 we would obtain an optimal solution to P when $T = \{2, 3\}$, i.e., when warehouse 1 is closed and warehouses 2 and 3 are open. Notice that cell (3, 8) has a cost of $-M$, thus in the optimal solution $x_{38} = S_3$, which has the effect of closing down warehouse. Cell (6, 1) has a cost of $-M$ which causes all of the demand in column 1 to be satisfied by the dummy factory, i.e., $x_{61} = S_3$. Thus setting $c_{38} = -M$ and $c_{61} = -M$ effectively closes down warehouse 1. On the other hand for warehouse

2, $c_{48} = M$ and $c_{62} = M$, so that $x_{48} = x_{62} = 0$. This causes the demand in column 2 to be satisfied from rows 1, 2, or 4. The shipments from rows 1 and 2 represent units being shipped from factories 1 and 2 to warehouse 2. The shipment from row 4 in column 2 represents the unused warehouse capacity, and it is charged at the proportional fixed charge rate. Notice that if cell (4, 2) contains x_{42} units then exactly $S_4 - x_{42}$ units can be used to ship from warehouse 2 to the demand centers. This is exactly the amount which is made available to warehouse 2 from the factories. Thus the conservation of flow equations (5) have been satisfied. Because a proportional amount of the fixed charge is assigned to both the used and unused parts of the warehouse capacity, the total fixed charge is covered in any feasible solution.

Fig. 4 contains an optimal solution to the $2 \times 5 \times 4$ examples solved in Section 6. In Fig. 4, $T = \{1, 3\}$ so that warehouses 1 and 3 are opened, and warehouses 2, 4 and 5 are closed. Factory 1 is not used at all so that $x_{1,10} = 61$. Factory 2 ships 21 units to warehouse 1, 40 units to warehouse 3, and 16 units directly to demand center 1. Warehouse 1 ships all 21 of the units it received from Factory 2 to demand center 3. The remaining unused 4 units of capacity at warehouse 1 are in cell (3, 1) and are charged at the proportional fixed charge rate. All of the flow for warehouses 2, 4, and 5 is in the last column, and their demand is satisfied entirely by the dummy factory. Warehouse 3 is used to capacity shipping 22 units to demand center 2, and 18 units to demand center 4. In this example

		Warehouses			Demand Centers				
Factories	{	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	c_{16}	c_{17}	A_1
		c_{21}	c_{22}	c_{23}	c_{24}	c_{25}	c_{26}	c_{27}	A_2
Warehouses	{	$\frac{f_3}{s_3}$	M	M	$c_{34} + \frac{f_3}{s_3}$	$c_{35} + \frac{f_3}{s_3}$	$c_{36} + \frac{f_3}{s_3}$	$c_{37} + \frac{f_3}{s_3}$	s_3
		M	$\frac{f_4}{s_4}$	M	$c_{44} + \frac{f_4}{s_4}$	$c_{45} + \frac{f_4}{s_4}$	$c_{46} + \frac{f_4}{s_4}$	$c_{47} + \frac{f_4}{s_4}$	s_4
		M	M	$\frac{f_5}{s_5}$	$c_{54} + \frac{f_5}{s_5}$	$c_{55} + \frac{f_5}{s_5}$	$c_{56} + \frac{f_5}{s_5}$	$c_{57} + \frac{f_5}{s_5}$	s_5
Dummy Factory		M	M	M	M	M	M	M	$\sum_{i \in I} s_i$
		s_3	s_4	s_5	d_4	d_5	d_6	d_7	d_8

Fig. 2.

each of the demand centers is supplied from a single warehouse. This is not the case in general.

The idea behind the cost operator approach to solving $P(T)$ is to open the warehouses in T and close those in $I - T$ by fixing in or out of the basis each of the cells $(m, 1)$ through (m, l) and $(q + 1, n)$ through $(q + l, n)$. For example the problem $P(I)$, in which all of the warehouses are open, can be obtained by solving the transportation problem shown in Fig. 2. Then to apply Rule 1 in Section 3 we use two cost operators for each $i \in I$. For example we would calculate $\Delta Z_3(I - \{3\}) = Z(I) - Z(I - \{3\})$ by fixing in cells $(3, 8)$ and $(6, 1)$. Doing this would yield a solution to the problem shown in Fig. 1, and permits the evaluation of Rule 1. In general, the addition of a warehouse to a given set or the deletion of a warehouse from a given set requires the fixing in or out of two cells. The amount of work needed to fix two cells in or out using cost operators is much less than the computational effort of solving a transshipment problem from scratch. Since the branch and bound search algorithm to be described in the next section requires the solution of transshipment problems for many sets T , the total computational effort saved by the fixing in and fixing out procedure is very large.

Also we should mention that in the case where $I' = \emptyset$, as in single stage problems, only one cost operator is needed to open or close a warehouse. In the single stage formulation columns 1 through l and rows 1 through 2 are absent. Thus in order to fix in or out warehouse i we need only apply a cost operator to cell (i, n) .

5. The cost operator algorithm

To describe the cost operator branch and bound algorithm we use the following notation:

- l = level of the search tree;
- Q = set of open warehouses;
- List(l) = list of warehouses which may be opened on level l ;
- Z^u = current upper bound;
- X = current best solution.

We begin by setting $l = 1$, $Q = \emptyset$ and $Z^u = Z(\emptyset)$ which is the (high) cost of supplying all demands from the factories directly. Then Rule 1 (see Section 3) is applied and we let $Q = I^*$. Then for each $i \in I - Q$ we apply the lower bound test (13) by setting $T = Q \cup \{i\}$, and checking to see whether

$$Z^u \geq Z(I) - \sum_{k \in I - T} \Delta Z_k(I - \{k\}). \quad (16)$$

If the right-hand side of (16) (which is a lower bound on the value of $Z(Q \cup \{i\})$) exceeds Z^u , then warehouse i can be permanently closed. For each i for which (16) holds true we apply cost operators to calculate $\Delta Z_i(Q)$. If $\Delta Z_i(Q) \geq 0$ then we permanently close warehouse i . Then we let List(1) be the set of all i such that $\Delta Z_i(Q) < 0$. (We place the warehouses in List(1) in order of non-increasing objective function values.) Next we remove the last warehouse, i , from List(1) and replace Q by $Q \cup \{i\}$. We let $Z^u = Z(Q)$, update X . If List(1) is empty, then we stop. Otherwise we calculate k^* as in (15). If $k^* = 1$, then we stop. Otherwise we replace l by $l + 1$ and continue to the

next level. At each level after the first, we perform the lower bound test (16) for each warehouse, i , in $\text{List}(l - 1)$ by letting $T = Q \cup \{i\}$. If warehouse i passes the test (16), we calculate $\Delta Z_i(Q)$. Again we let $\text{List}(l)$ be the set of warehouses such that $\Delta Z_i(Q) < 0$ in nonincreasing order. If $\text{List}(l)$ is not empty, we remove i , the last element of $\text{List}(l)$; let $Z^u = Z(Q \cup \{i\})$ and update X . If $\text{List}(l)$ is now empty, we backtrack by replacing l by $l - 1$, removing the last warehouse placed in Q and then continuing as before. Otherwise we calculate k^* as in (15). If $l \geq k^*$, we backtrack as just described. Otherwise we replace l by $l + 1$; replace Q by $Q \cup \{i\}$, and continue as before. The algorithm terminates when $\text{List}(1)$ has been emptied.

Now we formally state the *cost operator multistage location-allocation algorithm*

Step 1. Initialization. Set $l = 1$; $z^u = Z(\emptyset)$. Apply Rule 1. Let $Q = I^*$. For each $i \in I - Q$ apply test (16) letting $T = Q \cup \{i\}$. For each i which satisfies (16) calculate $\Delta Z_i(Q)$. Order those i such that $\Delta Z_i(Q) < 0$ in nonincreasing order and place them in $\text{List}(1)$. If $\text{List}(1)$ is empty, go to Step 5. Otherwise let k be the last warehouse in $\text{List}(1)$. Let $Z^u = Z(Q \cup \{k\})$; update X . Remove warehouse k from $\text{List}(1)$. If $\text{List}(1)$ is empty, go to Step 5. Otherwise go to Step 2.

Step 2. Bound by k^* . Calculate k^* as in (15). If $l \geq k^*$, go to Step 4. Otherwise replace l by $l + 1$; replace Q by $Q \cup \{k\}$ and go to Step 3.

Step 3. Forward branching. For each warehouse i in $\text{List}(l - 1)$ apply test (16) letting $T = Q \cup \{i\}$. For each i which satisfies (16) calculate $\Delta Z_i(Q)$. List those i such that $\Delta Z_i(Q) < 0$ in non-increasing order and place them in $\text{List}(l)$. If $\text{List}(l)$ is empty, go to Step 4. Otherwise let k be the last warehouse in $\text{List}(l)$. Let $Z^u = Z(Q \cup \{k\})$; update X . Remove warehouse k from $\text{List}(l)$. If $\text{List}(l)$ is empty, go to Step 4. Otherwise go to Step 2.

Step 4. Backtracking. Replace l by $l - 1$. Remove from Q the last warehouse placed in Q . If $l = 1$ and $\text{List}(1)$ contains only one warehouse go to Step 5. Otherwise let k be the last warehouse in $\text{List}(l)$. Remove k from $\text{List}(l)$. If $\text{List}(l)$ is empty, go to Step 4. Otherwise replace Q by $Q \cup \{k\}$; replace l by $l + 1$ and go to Step 3.

Step 5. Termination. Stop. The current solution X , is optimal.

Notice that each time Step 3 is performed a new list of feasible solutions to P is available. Many times these solutions have nearly optimal objective function values and thus they might be worthwhile to save.

After calculating I^* , the cost operator algorithm proceeds to open, one at a time, those warehouses which cause the largest decrease in the objective function value. This is continued until at level l , no further decrease in the objective function value is possible (i.e. $\text{List}(l)$ is empty). Then we move to the backtracking Step 4. The best feasible solution obtained by the cost operator algorithm before the first backtracking is called the *greedy solution*. We denote by Z^G the objective function value of the greedy solution. In [18], a *worst case* bound on Z^G was derived for submodular set functions which is,

$$\frac{Z^G - Z^*}{Z(\emptyset) - Z^*} \leq \left(\frac{|I^*| + k^* - 1}{|I^*| + k^*} \right)^{|I^*| + k^*}, \quad (17)$$

where Z^* is the optimal objective function value and k^* is obtained from (15).

In practice however, the actual percentage error obtained by the greedy solution is much smaller than the worst case bound. In Section 7 we will see that in most cases the greedy solution value is within 0.5 percent of optimality.

Finally we should point out that it would be trivial to 'reverse' the cost operator algorithm so that instead of starting with all of the warehouses closed, and opening them one at a time, we could start with all of the warehouses open and close them one at a time. This could be done by defining $Z'(T) = Z(I - T)$ in (1). $Z'(T)$ is also submodular, and results from Section 3 would remain valid for Z' . The reverse algorithm might be better suited to handle problems where an optimal number of warehouses tends to be close to the total number of potential warehouse sites.

6. Example

Fig. 3 shows the cost tableau for a $2 \times 5 \times 4$ multistage location problem. Fixed charges for warehouses 1–5 are: 150, 217, 200, 264 and 140. For an explanation of how the costs in the tableau are calculated, see Fig. 1 and Section 4. The diagram in Fig. 5 is a 5-dimensional hypercube whose nodes represent all of the possible open warehouse combinations. The search tree will be a subgraph of this hypercube. The number above each node is the value of an optimal solution to P when only those warehouses marked in circle of the

		Warehouses					Demand Centers					
Factories	{	8	5	7	5	9	37	28	33	26	0	61
		6	6	6	5	9	22	27	36	28	0	79
Warehouses	{	6	M	M	M	M	17 + 6	16 + 6	10 + 6	16 + 6	X	25
		M	8	M	M	M	18 + 7	14 + 7	17 + 7	14 + 7	X	31
		M	M	5	M	M	14 + 5	13 + 5	19 + 5	11 + 5	X	40
		M	M	M	11	M	13 + 11	12 + 11	11 + 11	13 + 11	X	24
		M	M	M	M	5	14 + 5	13 + 5	14 + 5	15 + 5	X	28
Dummy Factory		X	X	X	X	X	X	X	X	X	0	148
		25	31	40	24	28	16	22	21	18	211	

Fig. 3.

node are open. The number in the parenthesis above a node is the total cost lower bound for that node obtained by calculating (16).

We illustrate the steps of the algorithm applied to this example.

Step Calculations

- | | |
|--|--|
| <p>1 Set $l = 1$, $Q = \emptyset$. $Z^u = Z(\emptyset) = 2107$. Applying Rule 1 we get $\Delta Z_1(I - \{1\}) = 2303 - 2201 = 102$; $\Delta Z_2(I - \{2\}) = 217$; $\Delta Z_3(I - \{1 - \{3\}\}) = 316$; $\Delta Z_4(I - \{4\}) = 162$; $\Delta Z_5(I - \{5\}) = 140$. $l^* = Q = \emptyset$. For each $i \in \{1, 2, 3, 4, 5\}$ apply test (16). None of the lower bounds exceeds the current upper bound. (See the numbers in parenthesis in Fig. 5.) Next calculate $\Delta Z_1(\emptyset) = 1880 - 2107 = -227$; $\Delta Z_2(\emptyset) = -94$; $\Delta Z_3(\emptyset) = -123$; $\Delta Z_4(\emptyset) = -136$; $\Delta Z_5(\emptyset) = -105$. Let $\text{List}(1) = \{2, 5, 3, 4\}$. Let $Q = \{1\}$; $Z^u = 1880$; $l = 2$; go to Step 2.</p> <p>2 $Z(I) = 2334$; $\sum_{i \in I} f_i = 1002$; the fixed charge ordering is $f_5 = 140$, $f_1 = 150$, $f_4 = 200$, $f_3 = 264$. The DV ordering is $DV_2 = 0$, $DV_5 = 0$, $DV_4 = 38$, $DV_1 = 42$, $DV_3 = 48$. $Z^u = 1880$. Setting $k^* = 2$ we get from (19) $1332 + 290 + 38 \leq 1880 \leq 1332 + 490 + 0$. Since $l = 1 < 2$, let $l = 2$; let $Q = \{1\}$; go to Step 3.</p> <p>3 For each $i \in \{2, 5, 4, 3\}$ apply test (16) setting $T = \{1\} \cup \{i\}$. (See Fig. 5.) For each $i \in \{2, 5, 4, 3\}$ calculate $\Delta Z_i(\{1\})$. $\Delta Z_3(\{1\}) = -118$. $\Delta Z_4(\{1\}) = 32$; $\Delta Z_5(\{1\}) = 22$; $\Delta Z_2(\{1\}) = -18$. Let $Z^u = 1762$. $\text{List}(2) = \{2\}$. Go to Step 4.</p> | <p>Step 2 Calculations
Setting $k^* = 2$ we get from (19) $1332 + 290 + 38 \leq 1762 \leq 1332 + 490 + 0$. Since $l = 2 \geq 2$ go to Step 4.</p> <p>4 Let $l = 1$. Remove $\{1\}$ from Q (now $Q = \emptyset$). Let $k = 4$. Let $\text{List}(1) = \{2, 5, 3\}$. Let $Q = \{4\}$. Let $l = 2$; go to Step 3.</p> <p>3 For each $i \in \{2, 5, 3\}$ apply test (2) setting $T = \{4\} \cup \{i\}$. All solutions are fathomed (see numbers in parenthesis in Fig. 5). $\text{List}(2) = \emptyset$. Go to Step 4.</p> <p>4 Let $l = 1$. Remove $\{4\}$ from Q (now $Q = \emptyset$). Let $k = 3$. Let $\text{List}(1) = \{2, 5\}$. Let $Q = \{3\}$. Let $l = 2$. Go to Step 3.</p> <p>3 For each $i \in \{2, 5\}$ apply test (16) setting $T = \{3\} \cup \{i\}$. All solutions are fathomed (see Fig. 5). $\text{List}(2) = \emptyset$. Go to Step 4.</p> <p>4 Let $l = 1$. Remove $\{3\}$ from Q. (Now $Q = \emptyset$.) Let $k = 5$. Let $\text{List}(1) = \{2\}$. Let $Q = \{5\}$. Let $l = 2$; Go to Step 3.</p> <p>3 For $i \in \{2\}$ apply test (16) setting $T = \{5\} \cup \{i\}$. The solution is fathomed (see Fig. 5). $\text{List}(2) = \emptyset$. Go to Step 4.</p> <p>4 Let $l = 1$. Go to Step 5.</p> <p>5 Stop. $Z^u = 1762$ is optimal for warehouses $\{1, 3\}$. An optimal solution is shown in Fig. 4.</p> |
|--|--|

An optimal solution to the example is given in Fig. 4 when warehouses 1 and 3 are opened. The greedy

Warehouses					Demand Centers						
Factories	8	5	7	5	9	37	28	33	26	0 ⁶¹	61
	6 ²¹	6	6 ⁴⁰	5	9	22 ¹⁶	27	36	28	0 ²	79
Warehouses	6 ⁴	M	M	M	M	23	22	16 ²¹	22	M	25
	M	8	M	M	M	26	22	25	22	-M ³¹	31
	M	M	5	M	M	19 ⁰	18 ²²	24	16 ¹⁸	M	40
	M	M	M	11	M	24	23	22	24	-M ²⁴	24
	M	M	M	M	5	19	18	19	20	-M ²⁸	28
Dummy Factory	M	-M ³¹	M	-M ²⁴	-M ²⁸	M	M	M	M	0 ⁶⁵	148
	25	31	40	24	28	16	22	21	18	211	

Fig. 4.

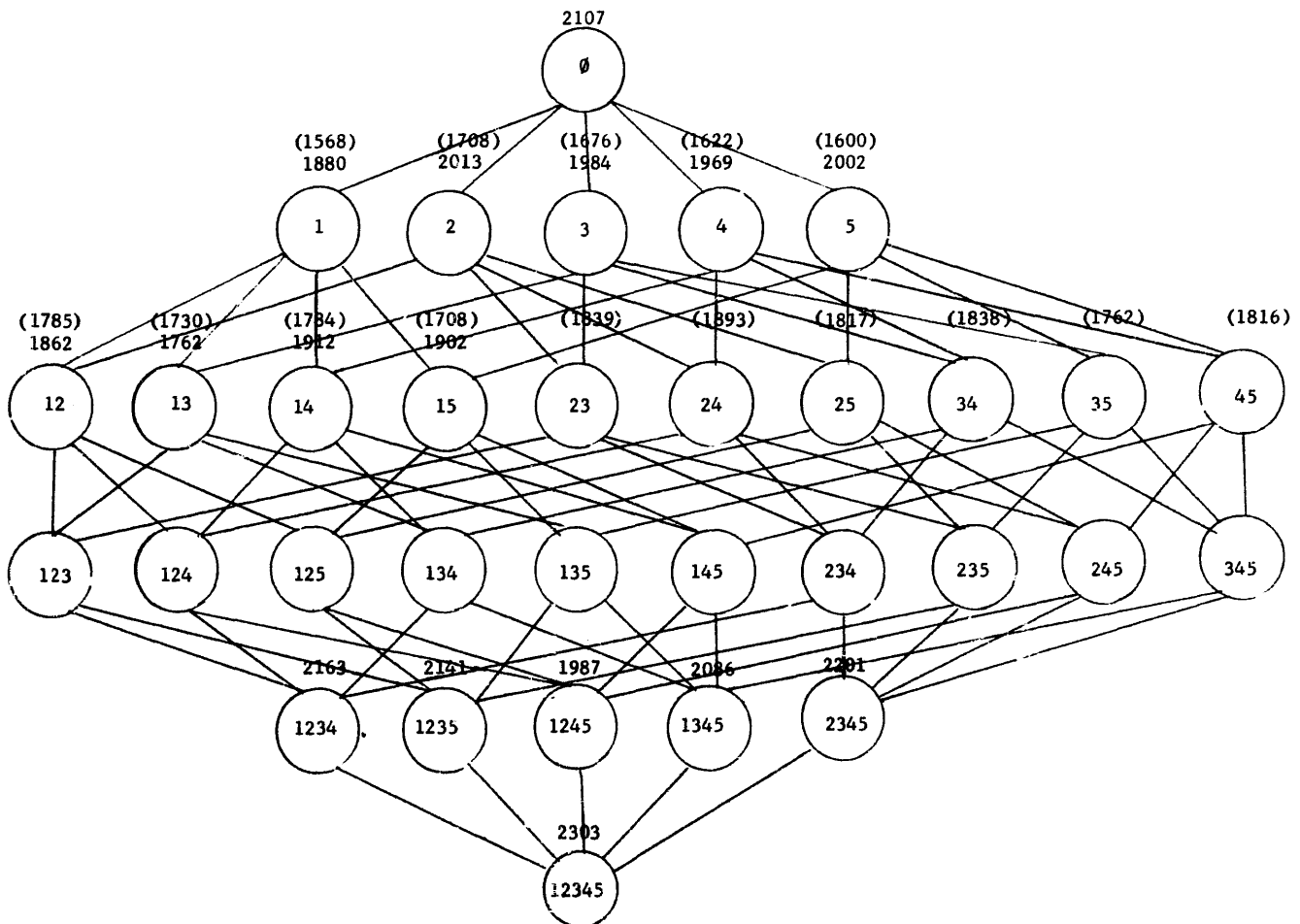


Fig. 5.

solution is optimal and is obtained at the circled vertex $\{1, 3\}$ in Fig. 5. The upper bound on an optimal number of warehouses, $k^* = 2$, was obtained in the first application of Step 2 and thus none of level three warehouse combinations were examined. In total, 16 of the vertices in Fig. 5 were generated. Six of the vertices on level 2 were fathomed using (16). The total number of transportation pivots required to solve the sample problem was 85.

7. Computational results

The cost operator algorithm of Section 5 was coded in FORTRAN IV and the runs were made on a

DEC 20 time sharing system. Many of the problems were run at different times during the day and thus the execution times may vary up to ten or fifteen percent depending upon the computing load of the machine. The maximum time allowed for solving any problem was 500 seconds.

All of the problems are derived from the Kuehn and Hamburger data which was originally presented in [14]. Problem sets I through VII are taken from Akinc and Khumawala [1], and VIII and IX are taken from Ellwein and Gray [8]. These are single stage problems of the type P1. Problem sets X through XV are multi-stage problems which are solved here for the first time.

Since it is necessary to read several articles to deter-

Table 1
Test problems

Problem Set	# Problems	(q×l×r)	$\sum d_i$	A_i	S_i	f_i
I	4	0×16×50	58268	-	5000	7,500/12,500/ 17,500/25,000/
II	1	0×16×50	58268	-	10000	17,500
III	4	0×16×50	58268	-	15000	7,500/12,500/ 17,500/25,000/
IV	4	0×16×50	58268	-	58628	7,500/12,500/ 17,500/25,000/
V	4	0×25×50	58268	-	5000	7,500/12,500/ 17,500/25,000/
VI	4	0×25×50	58268	-	15000	7,500/12,500/ 17,500/25,000/
VII	4	0×25×50	58268	-	58628	7,500/12,500/ 17,500/25,000/
VIII*	1	0×15×45	37440	-	1000 5000	15,000 40,000
IX*	1	0×15×45	37440	-	1500 7500	22,500 60,000
X	4	3×24×50	58268	35000	5000	7,500/12,500/ 17,500/25,000/
XI	4	3×24×50	58268	35000	15000	7,500/12,500/ 17,500/25,000/
XII	4	3×24×50	58268	35000	58628	7,500/12,500/ 17,500/25,000/
XIII	4	3×24×50	58268	30000	5000	7,500/12,500/ 17,500/25,000/
XIV	4	3×24×50	58268	30000	15000	7,500/12,500/ 17,500/25,000/
XV	4	3×24×50	58268	30000	58268	7,500/12,500/ 17,500/25,000/

* Numbers in the brackets correspond to ranges.

mine how the problems I through VII were originally created, we will describe them here. The Kuehn and Hamburger data represents a multistage system with 3 factories, 24 potential warehouse sites, and 50 demand centers located in the continental United States. Problems I through IV were formed by considering only the first 15 potential warehouse sites (i.e., Atlanta, Boston, through New Orleans as in [14]). To obtain the shipping costs (c_{ij}) from these sites to the 50 demand centers multiply each distance by \$ 0.025/mile unit cost, which is the bulk shipping rate. The sixteenth warehouse is actually the factory at Indianapolis. We used the distance from Indianapolis to the 50 demand centers multiplied by \$ 0.0125/mile unit as its shipping cost. The factory at Indianapolis has the same capacity as the other warehouses and it has a zero fixed charge. The problems V through VII are set up in a similar fashion except that all 24 potential warehouse sites are used. Problems X through XV are multistage problems which are also derived from the Kuehn and Hamburger data. All three factory sites were used. The factories have capacities of 30 000 or 35 000 units. To derive

the shipping costs, we multiplied the distances from each factory to each warehouse by \$ 0.0125/mile unit cost, and from each factory to each demand center by \$ 0.0375/mile unit cost. To get the shipping costs from the warehouses to the demand centers, we multiplied the distance by \$ 0.0250/mile unit cost.

Table 1 contains a description of all of the test problems. Table 2 contains the computational results for the single stage problems and Table 3 for the multistage problems. The percent error formula used to evaluate the greedy solution was,

$$\frac{Z^G - Z^*}{Z^*} \times 100,$$

where Z^G is the greedy value and Z^* is the optimal value. Also included in Table 2 are the execution times for the same problems solved in [1,17]. For test purposes the problem sets I through XV are very interesting because they contain problems with a wide range of difficulty.

The results for the greedy solution were amazing.

Table 2
Computational results

Problem Set	% error Greedy	# Vertices to find Optimal	Total # Vertices	$ I^* $	Total # Warehouse	Total # Pivots	N & T (DEC-20)	A & K IBM (370/165)	Nauss IBM (370/168)
I-1	0	25	25	11	13	163	1.61	10.21	7.6
I-2	0	21	21	11	12	152	1.43	9.15	1.5
I-3	0	21	21	11	12	156	1.46	9.26	6.8
I-4	0	21	21	11	12	156	1.45	9.58	10.8
II	.19	58	147	5	8	1712	6.16	19.68	6.6
III-1	0	28	32	9	11	158	1.02	.23	1.5
III-2	0	30	37	7	9	217	1.19	.43	1.7
III-3	0	51	102	4	7	858	3.37	38.65	5.3
III-4	0	41	158	3	5	2582	8.36	34.39	2.9
IV-1	0	28	31	9	11	115	.96	47.5	-
IV-2	0	27	31	7	9	149	1.03	.44	-
IV-3	0	38	41	3	5	218	1.2	.30	-
IV-4	0	30	30	3	4	133	.85	.15	-
V-1	.47	254	1303	11	17	11700	56.7	.23	18.4
V-2	.72	128	6278	9	14	96462	422.8	120*	18.1
V-3	-	-	-	-	-	-	500*	120*	9.1
V-4	-	-	-	-	-	-	500*	120*	73.4
VI-1	.10	67	360	10	15	1417	10.35	.75	2.2
VI-2	0	78	908	6	11	6952	31.8	7.75	9.2
VI-3	.19	982	2172	5	8	19420	79.9	120*	18.1
VI-4	.15	1299	16367	2	7	146900	495.2	120*	27.2
VII-1	0	62	238	10	14	978	5.5	.68	-
VII-2	0	69	408	6	11	2408	9.87	1.65	-
VII-3	.20	104	264	4	8	1922	6.72	1.34	-
VII-4	.06	75	161	2	4	1088	4.05	.46	-
VIII	0	60	804	5	11	6247	29.6	12.55	1.8
IX	3.7	92	1304	2	7	9367	41.6	4.17	8.1

* Computation terminated

- Not available

Table 3
Computational results

Problem Set	% error Greedy	# Vertices to Optimal	Total # Vertices	$ I^* $	Optimal # Warehouses	Total # Pivots	CPU Time (Dec-20)
X-1	0	56	143	11	15	2414	11.88
X-2	.30	514	858	8	13	12287	61.39
X-3	.24	277	680	6	11	7411	39.15
X-4	1.6	492	3627	1	7	39084	197.24
XI-1	0	54	99	9	13	1708	8.51
XI-2	0	71	521	5	8	7895	36.09
XI-3	0	62	184	5	8	2830	13.69
XI-4	0	90	837	2	6	14523	66.88
XII-1	.02	66	76	9	12	2874	17.69
XII-2	0	69	352	5	8	6885	34.43
XII-3	0	75	452	4	8	8431	41.26
XII-4	0	81	499	3	6	10383	50.1
XIII-1	0	56	145	11	14	2412	12.16
XIII-2	.30	514	858	8	13	12144	62.97
XIII-3	.24	277	680	6	11	7840	45.44
XIII-4	1.7	492	3627	1	7	41054	246.83
XIV-1	0	54	100	9	13	2060	12.85
XIV-2	0	71	537	5	8	8480	44.7
XIV-3	0	62	184	5	8	3152	19.9
XIV-4	0	90	838	2	6	14463	75.7
XV-1	.01	73	84	9	12	3627	23.8
XV-2	0	69	363	5	8	7813	44.3
XV-3	0	75	454	4	8	9187	49.8
XV-4	0	81	499	2	6	11234	60.53

The largest percent error incurred was 3.7 (see Table 2, IX), however in general the percent was less than 0.5 percent and the greedy solution was an optimal solution in 32 out of 51 problems.

Another interesting statistic is the number of vertices required to find the optimal solution as compared to the total number of vertices searched. An optimal solution is usually located very early in the search process and most of the effort is typically spent verifying its optimality. This experience is similar to that found in solving other kinds of integer programming problems.

The set I^* , which represents the warehouses fixed open in the initialization phase of the cost operator algorithm, often accounts for as much as 85% of the total number of open warehouses in an optimal solution. The uncapacitated problems (sets IV, VII, XII, XV) were very easy to solve as expected. As far as problem difficulty is concerned, those problems for which an optimal number of warehouses is approximately one half of the total number of potential warehouse sites are usually the most difficult. This is probably due to the fact that the number of possible warehouse combinations, $\binom{l}{l/2}$ when l is even or $\binom{l}{(l+1)/2}$ when l is odd, is the maximum of the binomial coefficients $\binom{l}{x}$.

The CPU times quoted in Table 2 were obtained

by three different sets of authors and are difficult to compare due to the differences in computers and in programming efficiency. As far as machine speeds is concerned, the IBM 370/168 is the fastest, followed by the IBM 370/165 and then by the DEC-20. However, actual speed ratio factors for pairs of the computing machines are virtually impossible to find. It seems fair to state that the performance of the three codes shown in the table are not significantly different; we may say they represent the state of the art of computational results on these problems.

Table 3 exhibits computational results on multistage problems having 3 factories, 24 warehouses, and 50 demand centers. As can be noted, the computation times are relatively small and exhibit a relatively small variance for integer programming problems. We again found the performance of the greedy solution to be even better than for the single stage problems. This is perhaps due to the fact that the factories have a large total capacity, and they can ship to customers directly if many of the warehouses are closed down.

8. Conclusions

We have presented a cost operator algorithm for solving multistage location-allocation problems which

does not employ problem relaxations as do the other currently best approaches [1,17]. Computational results indicate that this method is competitive with the others. The greedy solutions obtained by the method are usually extremely close to or are optimal. Also, because the method computes many near optimal solutions as it solves the problem, these near optimal solutions can be saved and printed out for use by a manager if he or she desires. Computational results on the solution of multistage location problems are presented here, but we have been unable to find other published results on such problems for comparison. The performance of our method on these problems is encouraging.

References

- [1] U. Akinci, and B.M. Khumawala, A branch and bound algorithm for capacitated warehouse location, *Management Sci.* 23 (6) (1977) 585–594.
- [2] D.A. Babayev, Comments on the note of Frieze, *Math. Programming* 7 (1974) 249–252.
- [3] O. Bilde and J. Krarup, Sharp lower bounds and efficient algorithms for the simple plant location problem, *Ann. Discrete Math.* (1977) 79–88 (published in Danish, 1967).
- [4] G. Cornuejols, Analysis of algorithms for a class of location problems, Technical Report No. 382, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY (1978).
- [5] G. Cornuejols, M.L. Fisher and G.L. Nemhauser, Location of bank accounts to optimize float: an analytic study and approximate algorithms, *Management Sci.* 23 (1977) 789–810.
- [6] P.S. Davis and T.L. Ray, A branch and bound algorithm for the capacitated facilities location problem, *Naval Res. Logist. Quart.* 16 (3) (1969) 331–343.
- [7] M.A. Efroymsen and T.L. Ray, A branch and bound algorithm for plant location, *Operations Res.* 14 (3) (1966) 361–368.
- [8] L.B. Ellwein and P. Gray, Solving fixed charge location-allocation problems with capacity and configuration constraints, *AIIE Trans.* 3 (1971) 290–299.
- [9] D. Erlenkotter, A dual based procedure for uncapacitated facility location, *Operations Res.* 26 (6) (1978) 992–1009.
- [10] A.M. Frieze, A cost function property for plant location problems, *Math. Programming* 1 (1971) 127–136.
- [11] A. Geoffrion and R. McBride, Lagrangian relaxation applied to capacitated facility location problems, *AIIE Trans.* 10 (1) (1977) 40–47.
- [12] A.M. Geoffrion and G.W. Graves, Multicommodity distribution system design by benders decomposition, *Math. Programming Study* 2 (1974) 82–114.
- [13] M. Guignard and K. Spielberg, A dual method for the mixed plant location problem, Technical Report No. 20 The Wharton School, University of Pennsylvania, Philadelphia, PA 19104 (1977).
- [14] A.A. Kuehn and M.J. Hamburger, A heuristic program for locating warehouses, *Management Sci.* 9 (9) (1963) 643–666.
- [15] D.I. Melnikov, The method of successive calculations and the plant location problem with nonfixed demands, *Work on Discrete Mathematics*, (Moscow 'Science' Publishers, Moscow, 1973).
- [16] S.C. Narula, U.I. Ogbu and H.M. Samuelson, An algorithm for the p -median problem, *Operations Res.* 25 (1977) 709–713.
- [17] R.M. Nauss, An improved algorithm for the capacitated facility location problem, *J. Operational Res. Soc.* 29 (12) (1978) 1195–1201.
- [18] G.L. Nemhauser, L.A. Wolsey and M.L. Fisher, An analysis of approximations for maximizing submodular set functions – I, *Math. Programming* 14 (3) (1978) 265–294.
- [19] G. Sa, Branch and bound and approximate solutions to the capacitated plant location problem, *Operations Res.* 17 (6) (1969) 1005–1016.
- [20] L. Schrage, Implicit representation of variable upper bounds in linear programming, *Math. Programming Study* 4 (1975) 118–132.
- [21] L.S. Shapley, Complements and substitutes in the optimal assignment-problem, *Naval Res. Logist. Quart.* 9 (1) (1962) 45–48.
- [22] K. Spielberg, Algorithms for the simple plant-location problem with some side conditions, *Operations Res.* 17 (1969) 85–111.
- [23] V. Srinivasan and G.L. Thompson, An operator theory of parametric programming for the transportation problem – I and II, *Naval Res. Logist. Quart.* 19 (1972) 205–252.